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On a sufficient condition for starlikeness

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Abstract

We give a sufficient condition for a normalized analytic function to be starlike.

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1. Introduction and Result

P. T. Mocanu [1] defined α-convex functions as follows: Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in the unit disc $\Delta=\{z:|z|<1\}$, with $f(z)f'(z)/z\neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α-convex in Δ if and only if the inequality

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0$$

holds in Δ. For the functions above, S. S. Miller, P. T. Mocanu and M. O. Read [2] proved the following theorem.

Theorem A. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is α-convex in the unit disc Δ, then $f(z)$ is starlike and univalent in Δ. Moreover, if $\alpha\geqq 1$, then $f(z)$ is convex for $|z|<1$, and if $\alpha\leqq -1$, then $1/f(1/z)$ is convex for $|z|>1$.*

In this paper, we partly improve Theorem A and to do so, we need the following lemma [3], [4].

Lemma A. *Let $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be analytic in the unit disc Δ and supposed that there exists a point $z_0\in\Delta$ such that*

$$\mathrm{Re}\{p(z)\}>0\ for\ |z|<|z_0|,$$

$$\mathrm{Re}\{p(z_0)\}=0\ and\ p(z_0)\neq 0.$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik$$

where

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \text{ when } \arg\{p(z_0)\} = \frac{\pi}{2}$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \text{ when } \arg\{p(z_0)\} = -\frac{\pi}{2}$$

where $p(z_0) = \pm ia$ *and* $a > 0$.

Theorem. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in the unit disc* Δ *and let us put*

$$F(z) = \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right].$$

Then, for the case $\alpha \geqq 0$ *or* $\alpha < -2$, *if* $F(z)$ *does not take pure imaginary value* l *where* $|l| \geqq \sqrt{(2+\alpha)\alpha}$, *then* $f(z)$ *is starlike in* Δ *and for the case* $-2 \leqq \alpha < 0$, *if*

$$\mathrm{Re}\{F(z)\} > 0 \text{ in } \Delta,$$

then $f(z)$ *is starlike in* Δ.

Proof. For the case $\alpha > 0$, from the hypothesis of the theorem we obtain $f'(z) \neq 0$ in Δ, because if there exists a point $z_0 \in \Delta$ such that

$$f'(z_0) = 0,$$

this contradicts the hypothesis of the theorem.

Let us put

$$p(z) = \frac{zf'(z)}{f(z)} \text{ and } p(z) \neq 0 \text{ in } \Delta.$$

Then it follows that

$$F(z) = p(z) + \alpha\frac{zp'(z)}{p(z)}.$$

If there exists a point $z_0 \in \Delta$ such that

$$\mathrm{Re}\{p(z)\} > 0 \text{ for } |z| < |z_0|,$$

$$\mathrm{Re}\{p(z_0)\} = 0 \text{ and } p(z_0) \neq 0,$$

then from Lemma A, for the case $\arg\{p(z_0)\} = \pi/2$, $p(z_0) = ia$ and $a > 0$, we have

$$F(z_0) = p(z_0) + \alpha\frac{z_0p'(z_0)}{p(z_0)} = ia + i\alpha k$$

and so, $F(z_0)$ is a pure imaginary value.

Then it follows that

$$\begin{aligned}\operatorname{Im}\{F(z_0)\} &= a + \alpha k \\ &\geqq \frac{1}{2}\left\{(2+\alpha)a + \frac{\alpha}{a}\right\} \\ &\geqq \sqrt{(2+\alpha)\alpha}.\end{aligned}$$

This contradicts the hypothesis of the theorem.

For the case $\arg\{p(z_0)\} = -\pi/2$, $p(z_0) = -ia$ and $a > 0$, we have also

$$F(z_0) = -ia + i\alpha k$$

and

$$\begin{aligned}\operatorname{Im}\{F(z_0)\} &= -a + \alpha k \\ &\leqq -a - \frac{1}{2}\alpha\left(a + \frac{1}{a}\right) \\ &= -\left\{(2+\alpha)a + \frac{\alpha}{a}\right\} \\ &\leqq -\sqrt{(2+\alpha)\alpha}.\end{aligned}$$

This is also contradicts the hypothesis of the theorem and it completes the proof of the case $\alpha > 0$.

For the case $\alpha < -2$, applying the same method and Lemma A, we can obtain the proof of the theorem.

Finally, for the case $-2 \leqq \alpha < 0$, it depends on [2].

References

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